BUSINESS CYCLE ANALYSIS WITH UNOBSERVABLE INDEX MODELS AND THE METHODS OF THE NBER

by

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I. Introduction

This paper studies relationships between the methods of business cycle description and analysis that were created by A.F. Burns and W.C. Mitchell at the National Bureau of Economic Research and versions of the one-unobservable index model of Sargent and Sims. In proposing the unobservable index model, Sargent and Sims were partly motivated by an intention to formalize the intuition underlying NBER methods. The present paper investigates how faithfully the unobservable index model reflects the NBER vision. We proceed by showing that when a one-unobservable index model is true, NBER methods can be interpreted as providing rough but sensible estimates of some of the index model's key parameters. We show how to read the statistics produced in applications of unobservable index models, and how to match them with related statistics produced by NBER methods. In doing this, our aim is to form a bridge between the broad body of empirical results achieved by the NBER, and the results attainable using index models.

Underlying the NBER methods was the vision that the natural unit for analysis was the business cycle, a recurring event of varying length and amplitude that is reflected in observations on many economic time series. It happens that realizations of a one index model assume the form of irregular but recurrent fluctuations that are imperfectly reflected, with differing amplitudes and phases, in a collection of time series. Time series on individual series are noise-corrupted observations on filtered versions of a single common signal or index. Each variable possesses its own filter which is applied to the common index, and which permits differences across variables in their amplitude, phase, and coherence with the common index. This common hidden index forms an intermediary through which occur all of the interdependencies among variables that characterize the business cycle. The presence of the variable-specific noises permits variability in the phases across different cyclical episodes. The one-dimensional nature of interdependencies and the irregular but recurrent nature of the fluctuations produced by a one-index model seem to us to be consistent with the vision that motivated Mitchell to organize observations around the concept of a reference cycle.

Burns and Mitchell defined a business cycle as follows:

> "...a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle; this sequence of changes is recurrent, but not periodic; in duration business cycles vary from more than one year to ten or twelve years; they are not divisible into shorter cycles of similar character with amplitudes approximating their own."
> (Source: A.F. Burns and W.C. Mitchell, Measuring Business Cycles, National Bureau, 1946, p.3.; Also quoted in W.C. Mitchell, What Happens During Business Cycles: A Progress Report, National Bureau, 1951.)

Mitchell describes the idea of observing many noise-corrupted indicators of a common underlying factor:

> "We dared not economize by relying solely upon index numbers and broad aggregates to represent the movements of prices, production, employment, and the like; for these broad summaries conceal highly significant differences in the cyclical behavior of their components. To represent important types of activity, such as producing durable goods or disbursing wages, we sought groups of series; by doing so we gained a better chance of reaching valid conclusions than if we depended on a single witness."
> (Mitchell, W.C., What Happens During Business Cycles, p.8)

This view of economic time series as unreliable or noisy witnesses to a common underlying phenomenon readily suggests a factor or hidden index model. Koopmans asserted that Burns and Mitchell's statistical procedures effectively assumed such a model:

> "The notion of a reference cycle itself implies the assumption of an essentially one-dimensional basic pattern of cyclical fluctuation, a background pattern around which the movements of individual variables are arranged in a manner dependent on their specific nature as well as on accidental circumstances. (There is a similarity here with Spearman's psychological hypothesis of a single mental factor common to all abilities.)"

These passages informally support the notion that an unobservable index model can be construed to underlie NBER methods. The calculations contained in this paper aim to make this notion more formal and concrete, and to provide experience in comparing results produced by the two sets of methods under study.

The remainder of this paper is organized as follows. In section 2 we describe the parameters of the unobservable index model that we regard as matching up most readily with those produced by NBER methods. In section 3 we describe our understanding of NBER methods and a computer program that we have written to automate those methods. There are a couple of points at which the NBER methods require the exercise of judgement. We identify these points, and reveal our method for mechanizing these judgements. Section 4 presents a simulation study in which our version of NBER methods is applied to artificial data generated by an unobservable index model. Particular statistics produced by the NBER methods are compared with particular parameters of the unobservable index model. We resort to a simulation study because the NBER methods produce a set of statistics that depend on the behavior of sample paths in a way that would be complicated to analyze. Section 5 reports the results from applying both NBER methods and the unobservable index model to a collection of postwar U.S. time series. Section 6 concludes the paper.

2. Analogues of Some NBER Statistics

2.a The Unobservable Index Model

A one-unobservable index model is[1]

(2.1) $$y_{it} = \lambda_i(L)f_t+\varepsilon_{it}, \quad i=1,\ldots,n$$

In (2.1) $\lambda_i(L)$ is a square summable, two-sided polynomial in the lag operator L. The unobservable random process $f_t,\varepsilon_{1t},\ldots,\varepsilon_{nt}$ satisfies $Ef_t=0$, $E\varepsilon_{1t}=E\varepsilon_{2t}=\ldots=E\varepsilon_{nt}=0$, and

(2.2)
$$Ef_tf_{t-s} = \begin{cases} 1 & s=0 \\ 0 & s\neq 0 \end{cases}$$

$$E\varepsilon_{it}\varepsilon_{jt-s} = \begin{cases} \sigma_i(s) & \text{when } j=i \\ 0 & \text{when } j\neq i \end{cases}$$

$$Ef_t\varepsilon_{it-s} = 0 \text{ for all } t,s,i,$$

where $\sigma_i(s)$ is a nonnegative definite sequence. Conditions (2.2) state that f_t is a serially uncorrelated process of unit variance which is uncorrelated with each ε_{it-s} for all s, and that the n ε_{it} processes are mutually uncorrelated, although each is permitted to be arbitrarily serially correlated. The variables y_{it}, $i=1,\ldots,n$ are observable. The force of (2.1) is that all of the cross correlations of $y_{1t},\ldots,y_{nt}$, including those at all leads and lags, are mediated through a common dependence on a single hidden index f_t.

By stacking the variables to form $y_t = (y_{1t},y_{2t},\ldots,y_{nt})^T$ and $\varepsilon_t = (\varepsilon_{1t},\varepsilon_{2t},\ldots,\varepsilon_{nt})^T$, we can write the model as

$$y_t = \sum_{u=-\infty}^{\infty} \lambda_u f_{t-u} + \varepsilon_t$$

with $\{\lambda_u\}_{u=-\infty}^{\infty}$ a sequence of n by 1 vectors satisfying $tr \sum \lambda_u \lambda_u^T < +\infty$. The restrictions (2.2) imply that

$$Ey_t y_{t-s}^T = \sum_{u=-\infty}^{\infty} \lambda_u \lambda_{u-s}^T + \sigma(s), \quad s=0,\pm 1,\pm 2,\ldots,$$

where $\sigma(s)$ is the n by n diagonal matrix with $\sigma_j(s)$ the j-th diagonal element. When n exceeds 2, hypothesis (2.2) embodies a restriction on the matrix covariogram of the vector process y.

Our hypothesis is that when a one-index model is true, Mitchell's methods can be interpreted as producing estimates that characterize important aspects of $\lambda_i(L)$ and $\sigma_i(s)$, in particular, the amplitude and phase of the Fourier transform of $\lambda_i(L)$, and the coherence of y_{it} with $\lambda_i(L)f_t$. To help motivate this hypothesis, imagine that the first $m < n$ variables y_{it} sum to an aggregate Y_t, like GNP or industrial production: $Y_t = \sum_{i=1}^{m} y_{it}$. This definition and (2.1) imply that

(2.3) $$Y_t = \lambda_Y(L) f_t + \varepsilon_{Yt} \quad \text{where } \lambda_Y(L) = \sum_{i=1}^{m} \lambda_i(L)$$

$$\varepsilon_{Yt} = \sum_{i=1}^{m} \varepsilon_{it}$$

It follows from the definition of ε_{Yt} and from (2.2) that ε_{Ys} is orthogonal to f_t for all t and s, so that Y_t fits the index model. We regard realizations of the random process $\lambda_Y(L)f_t$ as possessing many of the properties that Mitchell attributed to that unobservable whose irregular and recurrent fluctuations defined the "reference cycle".

The Fourier transform of $\lambda_j(L)$ is simply $\lambda_j(e^{-iw})$ or

$$(2.4) \qquad \lambda_j(e^{-iw}) = \sum_{k=-\infty}^{\infty} \lambda_{jk} e^{-iwk}$$

Represent both $\lambda_j(e^{-iw})$ and $\lambda_Y(e^{-iw})$ in the polar form

$$(2.5) \qquad \begin{aligned} \lambda_j(e^{-iw}) &= r_j(w) e^{i\theta_j(w)} \\ \lambda_Y(e^{-iw}) &= r_Y(w) e^{i\theta_Y(w)}. \end{aligned}$$

The responses of y_{jt} and Y_t to an input f_t of the form $f_t=\cos wt$ are given by (see Sargent [1979, p. 245]):

$$(2.6) \qquad \begin{aligned} \hat{y}_{jt}(w) &= r_j(w) \cos (wt+\theta_j(w)) \\ \hat{Y}_t(w) &= r_Y(w) \cos (wt+\theta_Y(w)). \end{aligned}$$

The amplitude and phase shift of the j^{th} variable relative to Y at frequency w_h are thus measured by

$$(2.7) \qquad \begin{aligned} \frac{r_j(w_h)}{r_Y(w_h)} &= \text{"amplitude relative to the aggregate"} \\ \theta_j(w_h) - \theta_Y(w_h) &= \text{"phase relative to the aggregate".} \end{aligned}$$

We interpret Mitchell's methods as estimating $r_j(w_h)/r_Y(w_h)$ and $\theta_j(w_h) - \theta_Y(w_h)$, or averages of them over a band of frequencies composing the business cycle.[2]

Given estimates of $\lambda_j(w_h)$, $\lambda_Y(w_h)$, we can generate several graphs that contain the same information about relative amplitudes and phases that Mitchell's nine-point graphs of reference cycle averages were intended to convey. We can simply plot realizations of the cosine waves given in (2.6). We can also plot realizations of $\hat{y}_{jt}(w_h)$ and $\hat{Y}_t(w_h)$ summed over a range of w_h's chosen to represent the business cycle:

$$\text{(2.8)} \qquad \hat{y}_{jt} = \sum_{h \epsilon B} r_j(w_h) \cos (w_h t + \theta_j(w_h))$$

$$\hat{Y}_t = \sum_{h \epsilon B} r_Y(w_h) \cos (w_h t + \theta_Y(w_h))$$

where $w_h = 2\pi h/T$, and B is a range of h's chosen to correspond to the business cycle.

The unobservable index model can be used to produce two other sorts of graphs that summarize the covariation of individual series with the hidden index. The first graph is of the projection of the unobservable $\lambda_j(L)f_t$ on observables y_{t-s}, $s=-r_1$, $,\ldots,-1,0,1,\ldots,r_2$ where y_t is the nx1 vector $(y_{it},\ldots,y_{nt})$. This projection can be represented as

$$\hat{E}[\lambda_j(L)f_t | y_{t-s}, s=-r_1,\ldots,r_2] = \sum_{k=-r1}^{r2} B_{jk} y_{t-k} \equiv B_j(L) y_t$$

Litterman and Sargent [1984] describe how to compute this projection given estimates of $\lambda_j(L)$ and $\sigma_j(s)$.

The second graph is of the projection of the cyclical components of $\lambda_j(L)f_t$ on observables, where the cyclical components correspond to a range of frequencies thought to correspond to the business cycle. This can be accomplished by modifying some of the computations described by Litterman and Sargent.[3]

2.b Some Cross-Spectral Statistics

Ordinary cross-spectral analysis can be used to generate an alternative set of statistics that are usefully compared to the statistics computed by Burns and Mitchell. Given a list of series $y_{jt}, j=1,\ldots,n$ with an aggregate $Y_t = \sum_{j=1}^{m} y_{jt}$, we propose to calculate the cross-spectra of each of y_{jt}; $j=1,\ldots,n$ with Y_t. To interpret these cross-spectra, consider the projection equations

$$y_{jt} = \sum_{k=-\infty}^{\infty} h_k^{(j)} Y_{t-k} + \varepsilon_t^{(j)} \qquad j=1,\ldots,n \tag{2.9}$$

where $E\varepsilon_t^{(j)} Y_{t-k} = 0$ for all k and all j. The projection (2.9) is well defined for any pair of jointly covariance stationary processes $\{y_{jt}, Y_t\}$. Let $\hat{E}(\cdot|\Omega_t)$ denote the linear least squares projection of $(\cdot)$ on an information set Ω_t. Now since $Y_t = \sum_{j=1}^{m} y_{jt}$, and since by the linearity of the projection operator $\hat{E}\left(\sum_{j=1}^{m} y_{jt}|\Omega_t\right) = \sum_{j=1}^{m} \hat{E}y_{jt}|\Omega_t$ for any information set Ω_t, it follows from (2.9) that

$$Y_t = \sum_{j=1}^{m} \hat{E}y_{jt} + \sum_{j=1}^{m} \varepsilon_t^{(j)} = \sum_{j=1}^{m} h^{(j)}(L)Y_t + \sum_{j=1}^{m} \varepsilon_t^{(j)} = Y_t.$$

Thus we have $\sum_{j=1}^{m} h^{(j)}(L) = I$, and $\sum_{j=1}^{m} \varepsilon_t^{(j)} = 0$. Since $\sum_{j=1}^{m} \varepsilon_t^{(j)} = 0$ by construction, it follows that the $\varepsilon_t^{(j)}$ processes are in general correlated across (j). Therefore, the list of projections (2.9) of y_{jt} on Y_t for $j=1,\ldots,n$ are not in general sufficient to characterize the cross-correlations between all pairs of y_{jt}'s for j chosen among $(1,\ldots,n)$. However, it can happen that most of the cross-covariances among the y_{jt} is accounted for by the $h^{(j)}(L)$'s. For example, if the coherences in projection (2.9) for most j=1,...,n are high, or if the dependencies across any pair of ε_{jt}'s are weak, then most of the covariance between pairs of y_{jt}'s can be inferred from the covariation of each with Y_t, which is summarized by the $h^{(j)}(L)$'s.

The $h^{(j)}$ of (2.9) are related to the cross-spectrum $S_{y_jY}(w)$ between y_j and Y by

(c) $$h^{(j)}(w) = \frac{S_{y_jY}(w)}{S_Y(w)}$$

where S_Y is the spectrum of Y and $h^{(j)}(w) = \sum_{k=-\infty}^{\infty} h_k^{(j)} e^{-iwk}$. We can express $h^{(j)}(w)$ in the polar form

(d) $$h^{(j)}(w) = \tilde{r}_j(w)e^{i\tilde{\theta}_j(w)}$$

We can also record the coherence between y_{jt} and Y_t, $\widetilde{coh}_j(w)$.

It is interesting to compare $\tilde{r}_j(w)$, $\tilde{\theta}_j(w)$, and $\widetilde{coh}_j(w)$ with the functions $r_j(w)/r_Y(w)$, $\theta_j(w)-\theta_Y(w)$ and $coh_j(w)$, respectively, which we calculate for the unobservable index model. If it happens that the one-index in the unobservable model is close to being a convolution of Y_t in the mean squared error sense, then each member of each of these three pairs of functions will closely resemble the other member of the pair.

Based on the cross-spectral analysis, it is possible to prepare graphs analogous to those described in equations (2.6) and (2.8) for the unobservable index model. For example, for the aggregate, at frequency w we would plot

(2.10) $$\tilde{Y}_t(w) = (S_Y(w))^{1/2}\cos(wt).$$

For y_{jt} at frequency w we would graph

$$(2.11) \qquad \tilde{y}_{tj}(w) = \tilde{r}_j(w)(S_Y(w))^{1/2}\cos(wt+\tilde{\theta}_j(w)).$$

We could plot (2.10) and (2.11) for any w chosen to be in the range of business cycle frequences. We could also plot sums of (2.10) and (2.11) for $w \epsilon B$, where B is a band of frequencies comprising the business cycle. Such graphs would be comparable to those described in (2.6) and (2.8).

The preceeding kind of cross-spectral analysis captures some but not all of the intuition motivating Burns and Mitchell's statistical procedures. It accomodates the idea that the "business cycle," which occurs with a time unit of varying length, determines the appropriate time unit for analyzing correlations and timing patterns among variables. It also captures the idea that the business cycle can be adequately characterized by a set of n graphs summarizing each of n series' pattern of variation over an average business cycle. However, the preceeding use of cross-spectral analysis fails to capture Mitchell's idea that by observing many noisy "witnesses" to the business cycle, one can gather a superior image of the reference cycle.

It is also of interest to compare Mitchell's procedures with the results of simply applying to the data on y_{jt} a low pass filter designed to isolate the business cycle frequencies, those between 12 years and one year, according to Mitchell. We desire to form

$$(2.12) \qquad y_{jt}(a,d) = b^{ad}(L)y_{jt},$$

where

$$b^{ad}(L) = \sum_{k=-\infty}^{\infty} b_k^{ad} L^k \tag{2.13}$$

$$b_k^{ad} = \frac{(\sin kd - \sin ka)}{\pi k} \text{ for integer } k.$$

The interval [a,d] is chosen to support the business cycle frequencies. For example, with quarterly data, using the formula period of cycle = $2\pi/w$, we would set $a=2\pi/48 = \pi/24$, $d=2\pi/4=\pi/2$ to isolate the band of frequencies between one and twelve years in length. The bandpass filter (2.13) is chosen to have Fourier transform

$$b^{ad}(e^{-iw}) = \begin{cases} 1 & w\in[-d,-a]\cup[a,d] \\ 0 & w\notin[-d,-a]\cup[a,d]. \end{cases}$$

Since the spectrum of $y_{jt}(a,d)$ equals the spectrum of y_{jt} times $|b^{ad}(e^{-iw})|^2$, application of the filter annihilates all variation outside of the business cycle frequencies.[4]

Unlike the statistics that we described earlier, bandpass filtering does not assume an index model structure. The bandpass filtering methods isolate the cyclical components of the sum $\lambda_j(L)f_t+\varepsilon_{jt}$, and make no attempt to purge the data of the cyclical components of the series-specific noises ε_{jt}. Thus, no particular vision of the structure of covariances between different series is imposed by this method.

3. Automating the NBER Methods

This section describes our understanding of Mitchell's method, and our computer program that automatically implements a version of that method. In this description, we assume the data are monthly.

Mitchell's method consists of the following steps, given a collection of time series x_{jt}, $t=1,\ldots,T$; $j=1,\ldots,n$. First, each series x_{jt} is seasonally adjusted, but not detrended. Second, turning points (peaks and troughs) in each series x_{jt} are detected and recorded over the period $t=1,\ldots,T$. These turning points are selected to correspond to recurrent fluctuations of length from one to ten or twelve years. They are called specific cycle turning points. Third, from the record of peaks and troughs in all n series, a list of turning points (peaks and troughs) in a "reference cycle" is determined. The concept of a reference cycle turning point is a "cluster of specific cycle turning dates." Judgement is used to weight the evidence from turning points in the individual series. The outcome of this step is a list of dates corresponding to reference cycle peaks $(p_1,\ldots,p_k)$ and dates $(\tau_1,\ldots,\tau_m)$ corresponding to reference cycle troughs. Fourth, the NBER nine-point graph is prepared for each series. The j^{th} reference cycle is defined as the period of time from τ_j to τ_{j+1}. Each reference cycle is divided into nine stages as follows (Mitchell, p. 14):

I – the three months centered at the initial reference cycle trough

V – the three months centered at the reference cycle peak

IX – the three months centered at the terminal trough

II, III, IV – formed by dividing the period between I and V into thirds

VI, VII, VIII – formed by dividing the period between V and IX into thirds.

These divisions assign each date t into one of the nine stages of a particular reference cycle. To complete Mitchell's procedure, perform the five steps that follow:

(a) Calculate the average value of series i within a reference cycle. That is, calculate

$$\bar{x}_i = \frac{1}{1+\tau_{j+1}-\tau_j} \sum_{t=\tau_j}^{\tau_{j+1}} x_{it}$$

for the i^{th} series. Mitchell calls this the "cycle base".

(b) Calculate the i^{th} series as a percentage of its average value for that reference cycle,

$$y_{it} = x_{it}/\bar{x}_i.$$

Mitchell calls y_{it} the "cycle relatives" of the i^{th} series for the particular reference cyle from τ_j to τ_{j+1}.

(c) Compute an average of the cycle relatives y_{it} over each of the stages I–IX. Call these $y_{i,I}^{(j)},\ldots y_{i,IX}^{(j)}$ for the i^{th} series and the j^{th} reference cycle (i.e., the reference cycle from τ_j to τ_{j+1}). The length of the j^{th} reference cycle in calendar time can be recorded as $T(j)=\tau_{j+1}-\tau_j+1$.

(d) Across all reference cycles in the sample (or sometimes a subset of reference cycles), average the measures for each of the nine stages. That is, compute

$$\bar{y}_{i,I} = \sum_{j=1}^{J} y_{i,I}^{(j)} / J$$

$$\vdots$$

$$\bar{y}_{i,IX} = \sum_{j=1}^{J} y_{i,IX}^{(j)} / J$$

troughs. For example, for quarterly data, we might define the indicator sequences

$$z_t = P(x_{jt}, x_{jt-1}, x_{jt-2}, x_{jt-3})$$

$$= \begin{cases} 1 \text{ if } x_{jt} \leq x_{jt-1} \leq x_{jt-2} \geq x_{jt-3} \\ 0 \text{ otherwise} \end{cases}$$

(3.1)

$$w_t = T(x_{jt}, x_{jt-1}, x_{jt-2}, x_{jt-3})$$

$$= \begin{cases} 1 \text{ if } x_{jt} > x_{jt-1} > x_{jt-2} \leq x_{jt-3} \\ 0 \text{ otherwise} \end{cases}$$

We would say that a peak occurred in x_{jt} at date t if and only if $z_{t+2}=1$. We would say that a trough occurred in x_{jt} at date t if and only if $w_{t+2}=1$. Modifications of the indicator functions P and T can be made for monthly data.

A second judgement is required for going from the list of dates for peaks and troughs in individual series to dates for peaks and troughs in reference cycles. Here we have experimented with two methods. The simpler involves merely accepting as dates for peaks and troughs the dates of peaks and troughs for one of the series that is best thought to measure aggregate activity. A well measured GNP series would be a natural candidate for this series. Mitchell explained that he did not use this method because he wanted to study periods and countries for which no reliable single measure of aggregate activity existed.

An alternative method involves formalizing Mitchell's notion of a "cluster of specific cycle turning dates". We have experimented with the following definition. A date t is said to be within a "clustering zone" of peaks and troughs if more than n/2 of our series have peaks at one of the dates (t-g,...,t-1,t,

$t+1,\ldots,t+g$). Then a reference cycle peak is defined as the modal date of peaks of these series having peaks within this clustering zone. A reference cycle trough is defined symmetrically.

To compare the results of Mitchell's method with those produced by the unobservable index model, we have formed two sets of statistics that are designed to summarize the patterns detected by the nine-point NBER graphs.

Let

$$z_i(\tau(R)) = \bar{y}_{i,R} \quad , \quad R=I,II,\ldots,IX$$

and where $\tau(R)$ maps Roman numerals I,...,IX into their Arabic counterparts 1,...,9. Thus $z_i(\tau)$, $\tau=1,\ldots,9$ is a record of the reference cycle averages for the nine points of the reference cycle for the i^{th} variable. We form the regression

$$(3.2) \qquad z_i(\tau) = a_i \cos w(\tau-1) + b_i \sin w(\tau-1) + \eta(\tau) \qquad \tau=1,\ldots,9$$

where $\eta(\tau)$ is a least squares residual orthogonal to both $\cos w(\tau-1)$ and $\sin w(\tau-1)$ for $\tau=1,\ldots,9$, and where $w=2\pi/8$. This regression can be represented as

$$(3.3) \qquad z_i(\tau) = \tilde{r}_i \cos(w(\tau-1)+\tilde{\theta}_i)+\eta(\tau)$$

where $\tilde{r}_i = (a_i^2+b_i^2)^{1/2}$ and $\tilde{\theta}_i = \tan^{-1}(-b_i/a_i)$. Let $\tilde{r}_Y$ and $\tilde{\theta}_Y$ be the analogous quantities computed for an aggregate series like GNP. Our intention is to compare $\tilde{r}_i/\tilde{r}_Y$ and $(\tilde{\theta}_i-\tilde{\theta}_Y)$ with the measures of relative amplitude and phase that are associated with the index model.

For our second set of statistics, we simply use the observations on averages over individual reference cycles.

$$z_i^{(j)}(\tau(R)) = y^{(j)}_{iR}.$$

We now run the regression

$$z_i^{(j)}(\tau) = \hat{a}_i \cos\ w(\tau-1) + \hat{b}_i \sin\ w(\tau-1) + \eta^{(j)}(\tau)$$
$$\tau=1,\ldots 9; \quad j=1,\ldots,J$$

where $\eta^{(j)}(\tau)$ is a least squares residual that is orthogonal to both $\cos\ w(\tau-1)$ and $\sin\ w(\tau-1)$. This regression can be represented as

$$z_i^{(j)}(\tau) = \hat{r}_i \cos(w(\tau-1)+\hat{\theta}_i)+\eta^{(j)}(\tau) \tag{3.4}$$

where $\hat{r}_i=(\hat{a}_i^2+\hat{b}_i^2)^{1/2}$ and $\hat{\theta}_i=\tan^{-1}(-\hat{b}_i/\hat{a}_i)$. The R^2 in this regression gives an indication of the variance of the phase and amplitude. We expect the R^2 for variable i to be comparable to the coherence between the i^{th} variable and the index.

The regressions (3.2) and (3.4) would be computed when detrended data were used. When trends are not removed from the data, we would add a linear trend to regressions (3.2) and (3.4).

4. Comparisons with Artificial Data

In this section we describe an experiment that is designed to study how the results of NBER methods match up with those of the one-index model under ideal conditions when we know a one unobservable index model to be true. We use a random number generator to produce a collection of artificial data that are drawn from a one-index model. We apply our automated version of NBER methods to these data, and compare the ordering of amplitudes and the relative phases across variables that are detected by the NBER nine-point reference cycle graphs with the population values for the index model of $r_j(w)/r_Y(w)$ and $\theta_j(w)-\theta_Y(w)$ for angular frequencies w in the range of business cycle frequencies described by Mitchell (one to twelve years).

The model is

$$y_{jt} = \lambda_j(L)f_t+\varepsilon_{jt} , \qquad j=1,\ldots,9$$

where f_t is a unit variance white noise that is orthogonal to ε_{jt-s} for all j and s. We set $y_{10t} \equiv Y_t = \sum_{j=1}^{6} y_{it}$. The series specific noises ε_{jt} are governed by

$$\varepsilon_{jt} = \frac{\lambda_j}{1-\rho L}\eta_{jt}$$

where η_{jt} is a white noise process with variance σ_j^2 and λ_j is the multiplier appearing in the numerator of $\lambda_j(L)$. We set $\lambda_j(L)$ as shown in Table 4.1. The amplitudes of the index components of variables 2 and 5 are approximately twice those of variables 1 and 4, while the amplitudes of the index components of variables

3 and 6 are approximately three times those of variables 1 and 4. The R^2 of variable j with the common index is approximately

$$R_j^2 = \left(1 + \frac{1-\lambda^2}{1-\rho^2}\,\sigma_j^2\right)^{-1}.$$

We set σ_j^2 so that the R_j^2 are those reported in Table 4.1. We set $\rho=.92$, and $\lambda=.98$ The coherence of variable j with the one index is

$$\mathrm{coh}_j(w) = \left(1+\frac{1+\lambda^2-2\lambda\cos w}{1+\rho^2-2\rho\cos w}\,\sigma_j^2\right)^{-1}.$$

For $\lambda > \rho > 0$, the coherence declines with increases in angular frequency w for $w > 0$.

Table 4.1

Parametrization chosen for experiment

j	$\lambda_j(L)$	R_j^2
1	$1/(1-.98L)$	.9
2	$2(.995)/(1-.98L)$	.75
3	$3(.995L^{2})/(1-.98L)$	.5
4	$1/(1-.98L)$	.3
5	$2(.995L^{-1})/(1-.98L)$	.9
6	$3(.995L^{-2})/(1-.98L)$	.5
7	$.5(995L^{3})/(1-.98L)$	.5
8	$.5(.995L^{-3})/(1-.98L)$	.9
9	$.01/(1-.98L)$	.05

Table 4.2a
Population values of Statistics of Interest

25-period cycle	Coherence	r_j/r_Y in percent	$\theta_j-\theta_Y$ in periods
VR11	.71	9.0	0.0
VR12	.44	18.0	-1.0
VR13	.21	26.9	-2.0
VR14	.10	9.0	0.0
VR15	.70	18.0	1.0
VR16	.21	26.9	2.0
VR17	.21	4.5	-3.0
VR18	.70	4.5	3.0
VR19	.01	0.1	0.0
VR20	.60		

Note: Variables 4,5,6 are mirror images of 1,2,3 in that the value of $r_j(\omega)/r_Y(\omega)$ for 4 (5,6) matches 1 (2,3), and that for $\theta_j(\omega)-\theta_Y(\omega)$ changes sign but remains the same in magnitude. Variable 8 is similarly a mirror image of 7.

Table 4.2b
Population values of Statistics of Interest

12 period cycle	Coherence	r_j/r_Y in percent	$\theta_j-\theta_Y$ in periods
VR11	.69	11.5	0.0
VR12	.42	22.9	-1.0
VR13	.20	34.7	-2.0
VR14	.10	11.5	0.0
VR15	.69	22.9	+1.0
VR16	.20	34.7	+2.0
VR17	.20	5.7	-3.0
VR18	.69	5.7	+3.0
VR19	.01	0.1	0.0
VR20	.46		

Table 4.2c
Population values of Statistics of Interest

8.1 period cycle	Coherence	r_j/r_Y in percent	$\theta_j-\theta_Y$ in periods
VR11	.69	20.2	0.0
VR12	.42	40.3	−1.0
VR13	.20	60.4	−2.0
VR14	.10	20.2	0.0
VR15	.69	40.3	+1.0
VR16	.20	60.4	+2.0
VR17	.20	10.1	−3.0
VR18	.69	10.1	+3.0
VR19	.13	0.2	0.0
VR20	.03		

Table 4.2d
Population values of Statistics of Interest

5.0 period cycle	Coherence	r_j/r_Y in percent	$\theta_j-\theta_Y$ in periods
VR11	.69	62.5	0.0
VR12	.42	124.4	−1.0
VR13	.19	186.6	−2.0
VR14	.09	62.5	0.0
VR15	.68	124.4	+1.0
VR16	.19	186.6	+2.0
VR17	.19	31.1	−3.0
VR18	.68	31.1	+3.0
VR19	.01	0.6	0.0
VR20	.03		

Table 4.3a

Index Model Results

25-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.78	.23	9.7
VR12	.48	- .40	22.0
VR13	.13	-3.14	20.1
VR14	.01	-2.96	10.5
VR15	.78	2.24	21.3
VR16	.25	1.54	30.8
VR17	.16	-2.43	4.5
VR18	.59	3.44	4.1
VR19	.04	-5.39	0.2

Table 4.3b

Index Model Results

12-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.76	.08	10.8
VR12	.47	- .57	20.7
VR13	.17	-1.92	24.9
VR14	.24	- .63	14.3
VR15	.77	1.00	22.1
VR16	.19	1.44	29.8
VR17	.24	-2.65	5.5
VR18	.60	3.10	4.7
VR19	.08	- .79	0.2

Table 4.3c

Index Model Results

8-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.66	− .43	15.5
VR12	.35	−1.27	28.9
VR13	.25	−1.76	47.1
VR14	.09	− .08	14.4
VR15	.66	.66	29.9
VR16	.30	1.72	63.2
VR17	.17	−3.57	7.4
VR18	.64	2.58	7.8
VR19	.05	− .66	0.3

Table 4.3d

Index Model Results

5-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.66	2.43	30.6
VR12	.40	1.44	54.1
VR13	.23	.38	118.8
VR14	.14	2.27	40.0
VR15	.71	-1.45	64.5
VR16	.20	- .45	109.4
VR17	.19	- .59	18.8
VR18	.57	.48	15.1
VR19	.00	- .86	0.2

Our specification is intended to make variables 1, 4, and 9 roughly in phase with the aggregate Y_t, while variables 2,3, and 7 lag the aggregate by approximately one, two, and three periods, respectively, and variables 5, 6, and 8 lead the aggregate by about one, two, and three periods, respectively. This interpretation stems from the observation that Y_t is composed as the sum of two contemporaneous, two leading, and two lagging series. The population parameter values are summarized in Table 4.1. The population values of coherence with the unobservable index, $r_j(w_h)/r_Y(w_h)$ and $\theta_j(w_h)-\theta_Y(w_h)$ for frequencies corresponding to 4-, 3-, 2- and 1-year cycles are recorded in Table 4.2. These parameters are also graphed for frequencies between 0 and π in Figure 4.8.

A single realization of 1500 terms was generated, and the first 500 terms discarded. The remaining 1000 points were aligned with a quarterly calendar beginning with 1850:1. Only part of the series used in the estimation are displayed in the graphs below.

Table 4.3 presents statistics from estimating the unobservable index model representation for the generated data. These numbers are to be compared to the population values recorded in Table 4.2. We see that the statistics we have chosen are fairly precisely estimated. The estimated coherences and gains are close to the population values, although at the higher frequencies, there is a tendency for the gain parameter to be underestimated. The estimated phase shifts are less accurate, however the signs and relative magnitudes do provide a reasonable picture of the underlying relations.

Figures 4.1a–4.1j, respectively, plot the raw data and band pass filtered data for variables y_{jt}, $j=1,...,9$ and Y_t; Y_t is the tenth variable reported. The top panel in each figure records the band pass filtered data, where the ideal band pass filter is chosen to include the range of frequencies $[a,b] = [2\pi/22, 2\pi/8]$. The bottom panel records the raw data.

Figures 4.2a–4.2j, respectively, plot the nine-point NBER reference cycle graphs for 15 of the 71 reference cycles identified in our data. These peaks and troughs were dated by applying the two consecutive downturn indicator function given in equation (3.1) to the aggregate variable Y_t. The graphs in Figure 4.2 are formed by adjoining the ending trough of one reference cycle to the beginning trough of the succeeding reference cycle. Inspection of these graphs reveals series 1, 4, and 7 as co-cyclical, series 2, 5, and 8 as lagging, and series 3, 6, and 9 as leading series, relative to the aggregate series Y_t, the tenth series.

Figures 4.3a–j show the average of reference cycle relatives of our ten series. These figures depict series 1–9 as having amplitudes, as measured by the maximum minus the minimum, of from 0.04 to 14.40. The aggregate series has an amplitude of 42.3; the amplitudes of the average cycles of series 1 through 6 sum to 42.4. These relative amplitudes agree well with those reported in Table 4.1 and 4.2. Also plotted in Figure 4.3 are one-third standard error curves on either side of the estimated average reference cyles.

Table 4.4 records the results of the regressions (3.2), and (3.4) calculated for the reference cycles identified in the preceding paragraph.

While on the whole the results in Table 4.4 are not too encouraging, some relevant features of the dynamics in the data are captured well by the regressions. The regressions do suggest that VR17, VR18 and VR19 have low coherence with the unobservable index insofar as the R^2's exhibited for equation (3.4) for these variables are approximately zero. The gains for these variables as measured by r_i/r_Y are also of the correct relative magnitude. Note that here, as in the estimated index model, the gain statistics provide an accurate picture of the underlying structure while the phase statistics are fairly imprecisely estimated. It is true that the $\theta_i-\theta_Y$ estimate for variables VR17 and VR18 are the largest in magnitude as they should be, and are of the correct sign but overall the column measuring phase shifts is fairly difficult to read.

Figures 4.4a-d record plots of $y_{jt}(w_h)=r_j(w_h)\ \cos(w_h t + (\theta_j(w_h)-\theta_Y(w_h))$ and of $\sum_{h\epsilon B} y_{it}(w_h)$ for j=1,2,3 and 10. In terms of their visual impression, these figures are to be compared to the NBER graphs of Figures 4.2 and 4.3. The graphs identify series 1 as being in phase, series 2 as lagging, and series 3 as leading.

Table 4.4

Burns-Mitchell Graph Regressions*
on artificial data

	R^2	$\theta_i - \theta_Y$ (radians)	r_i/r_Y (% of VR20)
VR11	.22 (.99)	.14 (−3.03)	3.5
VR12	.31 (1.0)	− .11 (− .19)	12.6
VR13	.45 (.99)	− .09 (− .14)	35.5
VR14	.18 (.98)	.04 (.06)	8.2
VR15	.15 (.97)	.34 (.45)	6.0
VR16	.47 (.99)	.05 (.10)	34.8
VR17	.02 (.82)	− .65 (− .97)	0.8
VR18	.03 (.88)	1.12 (1.20)	0.7
VR19	.00 (.08)	− .46 (1.02)	0.0
VR20	.66 (.99)		

*Results of regression (3.4) for all 71 reference cycles, with results of regression (3.2) for reference cycle averages in parentheses.

Figures 4.5a-i record the raw data for series 1-9, together with the projection $\hat{E}[\lambda_j(L)f_t|y_{t-s}, s=0,...,12]$. Figures 4.6a-i plot the raw data for each series against the projection of the cyclical part of $\lambda_j(L)f_t$ against y_{t-s}, $s=0,...,12$. In Figures 4.5 and 4.6, the projections are given by the dashed lines, while the solid lines depict the original series. The projections of the cyclical parts of $\lambda_j(L)f_t$ on y_{t-s} depicted in Figures 4.6 are smoother than the projection of $\lambda_j(L)f_t$ on y_{t-s} depicted in Figure 4.5. The projections in Figure 4.6 are fruitfully compared to the band-pass filtered versions in Figure 4.1, as well as to the back-to-back NBER reference cycles depicted in Figures 4.2.

For variables 1, 2, and 3, Figures 4.7a-c report graphs of a representative cycle formed by plotting one "cycle" of $\sum_{h \in B} y_{it}(w_h)$ against time. These graphs are intended to be compared with the NBER reference cycle averages of Figures 4.2. The ordering of variables by relative phase and amplitude are seen comparable.

For variables 1, 2, and 3, Figures 4.8a-c depict the population values of the spectrum of the index contribution $\lambda_j(L)f_t$, the gain $r_j(w_h)/r_Y(w_h)$, and the phase $\theta_j(w_h)-\theta_Y(w_h)$ as a function of $h=0,1,...T/2$ where $w_h=2\pi h/T$, $T=275$. Notice that the phase $\theta_j(w_h)-\theta_Y(w_h)$ is a linear function of w_h, with slope zero for $j=1$, -1 for $j=2$, and $+1$ for $j=3$.

For variables 1, 2, and 3 Figures 4.9 report the results of a cross-spectral analysis on the original raw data. The functions of frequency are plotted for w in $[0,\pi]$. Tables 4.5 and 4.6

record the cross-spectral analytic results comparable to that evidence presented in Tables 4.2 and 4.3. The reader should note that the column labelled coherence in these tables lists numbers for an object different from that for Tables 4.2 and 4.3. Here the coherence measure is taken with respect to the aggregate VR20, whereas the earlier tables recorded the coherence with the unobservable index. The rankings of the various variables by the gain statistic matches well with the true underlying relationships. At high frequencies, the gain statistics from ordinary cross spectral analysis underestimates the gain relations between the unobservable components. This is a version of an errors-in-variables result for a frequency domain regression although strictly speaking both the right- and left-hand side variables here are corrupted with noise.

Table 4.5a

Population cross spectral statistics

25-quarter Cycle	Phase (quarters)	Gain (% of GNP)
VR11	- .00	5.4
VR12	-1.00	10.7
VR13	-2.00	16.1
VR14	.00	5.4
VR15	+1.00	10.7
VR16	+2.00	16.1
VR17	-3.00	2.7
VR18	+3.00	2.7
VR19	- .00	0.0

Table 4.5b

Population cross spectral statistics

12-quarter Cycle	Phase (quarters)	Gain (% of GNP)
VR11	- .00	5.3
VR12	-1.00	10.5
VR13	-2.00	15.7
VR14	.00	5.3
VR15	+1.00	10.5
VR16	+2.00	15.7
VR17	-3.00	2.6
VR18	+3.00	2.6
VR19	.00	0.0

Table 4.5c

Population cross spectral statistics

8-quarter Cycle	Phase (quarters)	Gain (% of GNP)
VR11	- .00	4.3
VR12	-1.00	8.5
VR13	-2.00	12.8
VR14	- .00	4.3
VR15	+1.00	8.5
VR16	+2.00	12.8
VR17	-3.00	2.1
VR18	+3.00	2.1
VR19	- .00	0.0

Table 4.5d

Population cross spectral statistics

5-quarter Cycle	Phase (quarters)	Gain (% of GNP)
VR11	- .00	1.7
VR12	-1.00	3.4
VR13	-2.00	5.1
VR14	.00	1.7
VR15	+1.00	3.4
VR16	+2.00	5.1
VR17	-3.00	0.9
VR18	+3.00	0.9
VR19	.00	0.0

Table 4.6a

Estimated cross spectral analytic Results

25-quarter Cycle	Coherence*	Phase (quarters)	Gain (% of GNP)
VR11	.70	- .21	7.4
VR12	.54	- .58	14.1
VR13	.52	- .88	28.9
VR14	.15	1.27	7.4
VR15	.60	.74	13.8
VR16	.56	.51	30.3
VR17	.08	.40	1.9
VR18	.56	1.20	3.1
VR19	.10	-6.31	0.2

*Coherence refers to coherence of the variable with the aggregate VR20.

Table 4.6b

Estimated cross spectral analytic Results

12-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.35	− .12	5.4
VR12	.43	− .40	15.1
VR13	.38	− .87	27.2
VR14	.21	− .12	10.3
VR15	.47	.81	12.8
VR16	.48	.60	35.2
VR17	.07	−2.32	2.1
VR18	.34	2.81	2.7
VR19	.07	− .60	0.2

Table 4.6c

Estimated cross spectral analytic Results

8-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.35	- .40	6.8
VR12	.22	- .72	13.2
VR13	.31	- .85	30.9
VR14	.27	- .22	15.4
VR15	.40	.65	13.6
VR16	.37	.85	39.6
VR17	.03	-3.44	1.9
VR18	.33	2.51	3.2
VR19	.06	- .02	0.2

Table 4.6d

Estimated cross spectral analytic Results

5-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
VR11	.03	2.11	1.9
VR12	.04	- .27	5.0
VR13	.43	- .10	46.1
VR14	.11	.71	10.2
VR15	.00	-1.69	1.5
VR16	.45	- .01	45.6
VR17	.00	-1.48	0.5
VR18	.08	.26	1.7
VR19	.01	- .39	0.1

5. Results for Postwar U.S. Data

This section reports the results of applying NBER methods and a one-unobservable index model to a set of time series for postwar U.S. data for 1948:1 - 1982:4. The series analyzed included: consumption expenditure on durables, nondurables, and services, business fixed investment, residential investment, change in inventories, exports, imports, and government spending; these series are all in 1972 dollars, and sum to GNP. Thus in the notation of section 2, m=9, and $Y_t = \sum_{i=1}^{9} y_{it}$. We also included the following series: the GNP deflator, a quarterly average of the index of leading indicators, the three month Treasury bill rate, Standard and Poor's Stock Price Index, a measure of the real rate of interest, defined as the three month bill rate minus the ex-post rate of inflation in the GNP deflator, and the money supply, measured by M1.

Table 5.1a-d records the results from a one-index model for four frequencies of period 25, 12.5, 8.3 and 5 quarters. We report the coherence with the index, the "phase relative to the aggregate", and the "amplitude relative to the aggregate." The aggregate is taken to be GNP. Recall that the coherence gives an indication of the variance of the estimated relative phase and amplitude. The reader should note that relative phase and amplitude refer to the index component in each series and represent statistics from the projection of unobservables on unobservables. For brevity in the discussion, we will not always say "the index component in ..." but rather just refer to the original names of the noise corrupted versions themselves.

The figures for the gain of the index component in each series, on the component in aggregate output should be compared to the numbers in Table 5.1e. Table 5.1e lists the percentage of GNP that each series represents, as an average over the period 1948:1 to 1982:4. It shows the relative importance of the first 9 variables in terms of their dollar contribution to Gross National Product. The numbers also serve as a scaling factor to bear in mind for those series that do not directly enter the national income accounts but that various macroeconomic theories assert to be important in accounting for economic fluctuations. Total consumption constitutes over 60 percent of GNP, with the bulk of its contribution coming from expenditures on nondurables and services. Aggregate investment is on average 15 percent of GNP, while government purchases of goods and services over the sample period makes up 20 percent of GNP. Business fixed investment accounts for two-thirds of aggregate investment, with residential investment contributing the remaining one-third; inventory investment is trivial on average. In absolute value, international transactions as represented by exports and imports are about 10 percent of aggregate output.

The underlying dynamics of these variables as represented by their co-movements mediated through the one unobservable factor form a somewhat different picture. The relative importance of aggregate consumption and investment are now reversed. In the neighborhood of cycles of length approximately 4 years, the gains of the three measures of consumption on GNP sum to about 40 percent. For cycles of shorter periodicities, these figures fall to between 20 and 30 percent. The bulk of the contribution from aggregate consumption at these frequencies comes from consumption expenditures on durables, followed by consumption of nondurables and lastly by the consumption of services. At frequencies of between 1 and 4 years, the components of aggregate investment have a gain in total of about 60 percent. While inventory investment as a fraction of GNP has a mean that is insignificant, its underlying index component has a gain on the index component in GNP of over 30 percent at the 2 and 3 year cycles. Similarly the role of exports and imports is seen to be underemphasized if viewed strictly in terms of their mean contribution to GNP. On the other hand, while government spending is on average one-fifth of GNP, its index component has a gain on the index component in GNP of no more than 5 percent at the frequencies considered.

At 3 of the 4 frequencies analyzed, business fixed investment has the highest coherence with the unobservable index of any of the 9 series making up GNP. Government spending has a uniformly low coherence while consumption expenditure on durable goods has coherence of between 60 and 80 percent. At frequencies corresponding to cycles of between 2 and 3 years in length, the coherence of inventory investment with the unobservable index exceeds 70 percent. We see also that in the postwar period the coherence of exports and imports with the index is greater than 50 percent. At all frequencies coherence falls as we consider in turn consumption expenditure on durables, nondurables and finally services. The coherences of inventory and residential investment with the index are similar in magnitude and vary between 30 and 80 percent. Note that GNP itself has a coherence of approximately 80 percent with the unobservable index across frequencies. At the 2-, 3- and 4-year cycles, we see that the index of leading indicators has relatively high coherence (in excess of 0.85) with the unobservable factor. The price level has uniformly low coherence, as does our measure of the 3-month real interest rate. It is interesting that the coherence of M1 with the unobservable index rises with frequency: hence cycles of periodicities 2 years or less are associated more with movements in M1 than are the longer cycles. The 3-month Treasury bill rate has a coherence exceeding 0.60 at the 1-, 2- and 4-year cycles, as does the stock market at 2- and 3-year cycles.

At all frequencies considered, the stock market, the index of leading indicators and inverse movements in the 3-month Treasury bill rate and the expost real rate all have factor components that lead the factor component in GNP. (Note however that this does not imply that they help to predict GNP.) The GNP deflator lags at all frequencies, while M1 has a 1-quarter lead at the 1- , 2- and 3-year cycles. Consumption expenditure on durables and residential investment lead at all frequencies, while except for 4-year cycles, government spending lags.

Table 5.1b

Index Model Results

12.5-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
Consumption of Durables	.81	1.0	17.5
Consumption of Nondurables	.56	.76	9.9
Consumption of Services	.56	.18	5.0
Business Fixed Investment	.94	-.95	23.0
Residential Investment	.73	1.38	18.3
Inventory Investment	.73	.17	27.1
Exports	.50	-2.19	13.1
Imports	.61	-.43	9.7
Government Spending	.08	-5.58	6.7
GNP deflator	.11	-3.70	1.4
Index of Leading Indicators	.86	1.00	21.4
3-month Treasury Bills	.45	4.91	3.5
S&P 500 Stock Price Index	.68	2.28	31.2
Ex post 3-month real rate	.09	5.87	3.3
Money	.33	1.09	6.4
GNP	.88		

Table 5.1c

Index Model Results

8.3-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
Consumption of Durables	.75	.66	15.9
Consumption of Nondurables	.21	.03	5.7
Consumption of Services	.16	.84	2.5
Business Fixed Investment	.92	-.50	19.3
Residential Investment	.74	.99	17.0
Inventory Investment	.76	.24	33.3
Exports	.54	-1.35	12.6
Imports	.58	-.79	12.8
Government Spending	.02	-3.17	2.7
GNP deflator	.04	-.51	.6
Index of Leading Indicators	.86	1.02	19.7
3-month Treasury Bills	.70	4.95	4.0
S&P 500 Stock Price Index	.64	1.51	30.0
Ex post 3-month real rate	.18	5.65	5.9
Money	.62	1.06	10.4
GNP	.87		

Table 5.1e

Mean of Data as Percentage of Aggregate GNP

	Percent
Consumption of Durables	8.05
Consumption of Nondurables	26.76
Consumption of Services	26.57
Business Fixed Investment	9.95
Residential Investment	4.41
Inventory Investment	0.69
Exports	6.41
Imports	5.08
Government Spending	22.23
GNP deflator	9.57
Index of Leading Indicators	9.91
3-month Treasury Bills	0.43
S&P 500 Stock Price Index	7.12
Ex post 3-month real rate	0.03
Money	21.10

Table 5.2

Burns-Mitchell Graph Regressions[1]

	R^2	$\theta_i - \theta_Y$ (radians)	r_i/r_Y (% of GNP)
Consumption of Durables	.27 (.98)	.24 (−2.84)	21.8
Consumption of Nondurables	.27 (.99)	.23 (.13)	14.1
Consumption of Services	.08 (.98)	.91 (.04)	7.2
Business Fixed Investment	.36 (.99)	− .20 (− .87)	20.6
Residential Investment	.20 (.97)	.38 (.64)	17.5
Inventory Investment	.34 (.97)	− .16 (.31)	25.5
Exports	.05 (.95)	−2.03 (−2.04)	5.4
Imports	.15 (.93)	.17 (− .32)	9.1
Government Spending	.07 (.88)	− .51 (−1.23)	12.0
GNP deflator	.18 (.58)	2.83 (−2.25)	0.04
Index of Leading Indicators	.27 (.98)	0.28 (0.47)	21.0
3-month Treasury Bills[2]	.28 (.94)	2.20 (1.57)	2.5
S&P 500 Stock Price Index	.10 (.91)	.49 (.76)	15.1
Ex post 3-month real rate[2]	.12 (.65)	1.81 (1.28)	0.03
Money	.06 (.77)	1.24 (−1.28)	0.1
GNP	.40 (.99)		

[1]Results of regression (3.4) for all reference cycles, with results of regression (3.2) for reference cycle averages in parentheses.

[2]The statistic $\theta_i - \theta_Y$ is calculated relative to inverse movements in interest rates.

The results of applying NBER methods to linearly detrended data are summarized in Table 5.2. Regressions (3.2) and (3.4) are the vehicles used to summarize the results. The ordering of variables by amplitude are quite similar to the results from the index model. Business fixed investment is identified as a lagging series; while the stock market, the index of leading indicators and inverted movements in 3-month interest rates lead. There is some tendency for the ordering of variables by R^2 to match the ordering given by the coherences.

We applied Wecker's indicator function (3.1) to GNP to date peaks, both for detrended and non-detrended data, over 1948:1 - 1982:4. For the non-detrended data, this procedure identified five downturns at 1953:2, 1957:3, 1960:1, 1969:3, and 1974:2. For the data from which a linear least squares trend was removed, the procedure spotted ten downturns at 1951:3, 1953:1, 1955:3, 1959:2, 1962:2, 1964:2, 1966:4, 1969:1, 1973:1, and 1978:4. With the detrended data, more cycles are identified, and those identified in common with the non-detrended data are also dated earlier.

Figure 5.1 plots the nine-point NBER graphs for all of the reference cycles for the detrended data. Figure 5.2 reports the averages over all ten reference cycles. In Figure 5.1, notice that the fifth, sixth and seventh cycles, which are the cycles with peaks in 1962:2, 1964:2, and 1966:4, are mild in amplitude compared both with earlier and later cycles. (Their mildness is the reason that these cycles are not spotted if we do not

Table 5.1a

Index Model Results

25-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
Consumption of Durables	.80	1.27	18.5
Consumption of Nondurables	.77	.47	13.1
Consumption of Services	.40	-.82	9.0
Business Fixed Investment	.90	-1.75	26.8
Residential Investment	.61	4.10	19.1
Inventory Investment	.34	2.26	12.8
Exports	.48	-3.92	16.7
Imports	.75	-1.20	11.5
Government Spending	.02	12.47	4.4
GNP deflator	.17	-7.74	4.9
Index of Leading Indicators	.93	2.49	20.4
3-month Treasury Bills[1]	.64	10.45	4.1
S&P 500 Stock Price Index	.16	5.86	13.8
Ex post 3-month real rate[1]	.09	11.93	2.1
Money	.19	-1.13	12.2
GNP	.80		

[1]Phase is calculated relative to inverse movements in interest rates.

Table 5.1d

Index Model Results

5-quarter Cycle	Coherence	Phase (quarters)	Gain (% of GNP)
Consumption of Durables	.60	.37	21.1
Consumption of Nondurables	.32	.05	10.1
Consumption of Services	.10	.68	2.5
Business Fixed Investment	.38	.39	10.5
Residential Investment	.49	.57	11.8
Inventory Investment	.27	-.74	20.9
Exports	.82	-.11	20.2
Imports	.71	-.09	16.3
Government Spending	.04	-.68	3.9
GNP deflator	.002	-.46	.1
Index of Leading Indicators	.48	.75	9.5
3-month Treasury Bills	.64	2.57	7.3
S&P 500 Stock Price Index	.17	1.74	18.8
Ex post 3-month real rate	.11	3.01	6.1
Money	.78	1.11	13.7
GNP	.79		

detrend.)

From Figure 5.2, one gets a picture of relative phases and amplitudes that in our opinion agrees well with that given in Table 5.1. One finds durables consumption leading, the consumption of services lagging, business fixed investment lagging, residential investment leading, changes in inventories about contemporaneous, imports and exports lagging, the stock market, the index of leading indicators and inverted movements in the 3-month Treasury bill rate series leading.

Figures 5.3 and 5.4 report the results for the non-detrended data. The pattern of leads and lags is somewhat harder to read, than with the detrended data, but the pattern is similar.

We have also prepared graphs analogous to those in Section 4 that are obtained from an index model fitted to post-war US data. These would be read in the same way as we did in Section 4 in light of the lessons of that section.

Figure 5.5a-o and 5.6a-o display the detrended data together with the projections and band-pass projections respectively, calculated from the estimated index model. Here the band-pass filter supports periods of 5 to 25 quarters. These should be compared to the NBER reference cycles plotted in Figure 5.1.

Figure 5.7a-o contain various statistics on the index components in each series in relation to that in the aggregate. (The reader will recall that the appropriate inverse tangent function is determined only up to plus or minus $2k\pi$; we report the value in $[-\pi,\pi]$ in Table 5.1 while we plot a "smooth" branch in Figure 5.7). Those functions of frequency are plotted for w_h, $h = 0,1,\ldots,12$, $w_h = 2\pi h/25$. Figures 5.8a-o plot analogous 9-

point graphs obtained from the index model. The reader should compare these to our automated Burns-Mitchell 9-point graphs in Figure 5.2

6. Conclusion and Extensions

This study began partly as an attempt to compare the methods of analysis of Burns and Mitchell with the methodology and assumptions embodied in the unobservable index model of Sargent and Sims. One is a nonparametric nonstatistical approach to the collection of facts about economic time series, the other imposes a statistical discipline on the researcher but may also be liberally viewed as a nonparametric study of time series. Here "parameter" may be used in the sense of "a recognizable construct in the paradigm imposed by economic theory". In most instances, researchers interpret this to mean preferences or technology and take those to be the meaningful quantities for which to recover estimates. On another interpretation a parameter is any reliable object around which to organize patterns of quantitative thinking about processes that are of interest to the researcher. One property it must have to be useful in that role is that there must exist an easily understandable relation between itself and observables. That is, assigning a numerical value to a parameter must inform the researcher as to the likely dynamic behavior of the variables that he can observe. This serves a dual role in that it allows identification of further possible and necessary refinements in a theory, and it permits the analyst to predict and understand the evolution of the process under study. Intimately tied to this requirement is that the parameter be a reliable (that is, stable) guide to the relations of interest. Very generally, misspecification of these relations will invalidate the reliability of any parametrization that the researcher has chosen. It may be argued that any chosen representation for modelling data can only be an approximation and therefore that all modelled relations are misspecified to some extent.

In this work, we have chosen to organize our quantitative studies around a parametrization that is intuitive to the economist who has spent time thinking about the dynamic relations in observable macroeconomic variables. In doing so we have drawn upon tools that scientists in diverse fields have found useful and for which there is a developed statistical theory. We have shown the connections (or lack thereof) between the statistics implicit in an index model parametrization for macroeconomic variables and the calculations performed by Burns and Mitchell, and more generally associated with NBER studies on business cycles. The point is not merely one of methodology, however. For those who find the NBER calculations to be insightful, the parametrization that the index model represents must be similarly helpful towards the understanding of business cycle phenomena. Further the standardization obtainable with a method of analysis that is easily duplicated will help sharpen debate on those features of the business cycle that macroeconomists wish to understand. For these and other reasons the empirical results presented in this paper are useful.

One possible extension in this work is a more careful study of the distribution of the estimated parameters we presented in Section 5; we have tried to argue in the earlier part of this section that these are aids to our intuition in organizing quantitative thinking about the business cycle. The precision of the statements that can be made will be strengthened with greater knowledge of the uncertainty surrounding our parameter estimates.

A further development would be to exploit further the uses to which an estimated index model could be applied. By known methods of factoring spectral density matrices, one could map out the vector moving average representations of the variables of interest in the factor component. Assuming the validity of an index representation, these would provide similar insights into the dynamic interrelationships between variables as provided by impulse response functions calculated from vector autoregressive representations for data. The parsimonious parametrization provided by the index model representation would also be useful for forecasting exercises. The same ideas presented here are directly applicable also to a study of international macroeconomic variables, or the dynamic behavior of different asset prices, or the breakdown of factor incomes.

Some of these applications have already been pointed out and used in earlier work. Others are currently topics on our research agenda.

FOOTNOTES

1. The k-unobservable index model is

$$y_t = \lambda(L)f_t+\varepsilon_t$$

where y_t is an (nx1) vector, f_t is a kx1 vector of mutually orthogonal white noises, ε_t is an (nx1) vector of mutually uncorrelated but possibly serially correlated random processes, and $\lambda(L)$ is an (nxk) matrix of square-summable, possibly two-sided polynomials in the lag operator L. The generalizations of conditions (2.2) are

$$Ef_tf_{t-s}^T = \begin{cases} I & s=0 \\ 0 & s\neq 0 \end{cases}$$

$$Ef_t\varepsilon_{t-s}^T = 0 \qquad \text{for all } t,s$$

$$E\varepsilon_{it}\varepsilon_{jt-s} = \begin{cases} \sigma\ (s) & \text{when } j=i \\ 0 & \text{for all } t,s \text{ when } j\neq i \end{cases}$$

where $\sigma_i(s)$ is again a nonnegative definite sequence.

2. Consider the one-index model $y_t=\Lambda(L)f_t+\varepsilon_t$ where y_t is nx1, $\Lambda(L)$ is (nx1), $Ef_t^2=1$. For n > 2, $\Lambda(e^{-iw})\ \Lambda(e^{+iw})^T$ is identified. To identify $\Lambda(e^{-iw})$ an additional normalization is required because given any particular $\hat{\Lambda}(e^{-iw})$ satisfying $\hat{\Lambda}(e^{-iw})\hat{\Lambda}(e^{+iw})^T=\Lambda(e^{-iw})$ $\Lambda(e^{+iw})$, there exists a class of transformations $\Lambda^*(e^{-iw})$ of $\hat{\Lambda}(e^{-iw})$ that satisfy $\Lambda^*(e^{-iw})\ \Lambda^*(e^{+iw})^T = \Lambda(e^{-iw})\ \Lambda(e^{+iw})^T$. In particular, let $\Lambda^*(e^{-iw})=\hat{\Lambda}(e^{-iw})e^{-i\phi(w)}$, where $\phi(w)$ is any real valued scalar function of w. To identify $\Lambda(e^{-iw})$ from $\Lambda(e^{-iw})$ $\Lambda(e^{+iw})^T$, we impose the normalization that the last row $\lambda_n(e^{-iw})$ of $\Lambda(e^{-iw})$ be <u>real</u>. (This makes $\lambda_n(L)$ two-sided and